Spherical Function Representations: a Practical Survey
Supplemental 1/2: Formulas

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Abstract

This supplemental document shows the derivation of some formulas in the paper and tables of evaluated polynomial bases.

1. Derivation of Parameters using Log-Likelihood for EM-fitting

In Section 4.5 of the paper the Expectation Maximization algorithm is introduced to fit component mixture models robustly. In this section the derivation of exponential parameters is shown. Therefore, we maximize the log-likelihood

$$\mathcal{L}(i) = \sum_{s=1}^{M} f_s h_{si} \log p(\vec{d}_s | i),$$

with respect to one of the parameters. I.e. we search $\partial\mathcal{L}/\partial\lambda = 0$ for isotropic models and $\partial\mathcal{L}/\partial\alpha = 0$, $\partial\mathcal{L}/\partial\beta = 0$ for anisotropic models respectively. The notation is the same as in the paper:

$\mathcal{S}$	The set of samples with $\lvert\mathcal{S}\rvert = M$
$s \in [1, M]$	Index of the sample
$i \in [1, N]$	Index of the component or reference to its parameter set
$\vec{d}_s, f_s$	Observed direction and function value of sample s
$\vec{c}_i$	Central component direction; often μ in EM descriptions
$\vec{t}_i, \vec{b}_i$	Tangent and bitangent of anisotropic kernels
$\lambda_i, \alpha_i, \beta_i$	Isotropic or anisotropic frequency parameter of component i
w_i	Scale of a component (weight of the basis function)

Most times we express the kernels in terms of angles which depend on the following dot products:

$$\cos\theta = (\vec{d}_i \cdot \vec{c}_i)$$
$$\cos\phi = \sqrt{\frac{(\vec{d}_i \cdot \vec{t}_i)^2}{1-(\vec{d}_i \cdot \vec{c}_i)^2}} = \sqrt{\frac{(\vec{d}_i \cdot \vec{t}_i)^2}{(\vec{d}_i \cdot \vec{t}_i)^2 + (\vec{d}_i \cdot \vec{b}_i)^2}}$$
$$\sin\phi = \sqrt{\frac{(\vec{d}_i \cdot \vec{b}_i)^2}{1-(\vec{d}_i \cdot \vec{c}_i)^2}} = \sqrt{\frac{(\vec{d}_i \cdot \vec{b}_i)^2}{(\vec{d}_i \cdot \vec{t}_i)^2 + (\vec{d}_i \cdot \vec{b}_i)^2}}$$

1.1. Isotropic Cosine Distribution

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\sum_{s=1}^{M} f_s h_{si} \log\left(\frac{\lambda+1}{2\pi}\cos^{\lambda}\theta\right) &= \frac{1}{\lambda+1}\sum_{s=1}^{M} f_s h_{si}(1+(\lambda+1)\log\cos\theta) \\
&= \frac{\sum_{s=1}^{M} f_s h_{si}}{\lambda+1} + \sum_{s=1}^{M} f_s h_{si}\log\cos\theta \\
= 0 \Rightarrow \lambda &= \frac{\sum_{s=1}^{M} f_s h_{si}}{-\sum_{s=1}^{M} f_s h_{si}\log\cos\theta} - 1
\end{aligned}$$

1.2. Anisotropic Cosine Distribution

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\sum_{s=1}^{M} f_s h_{si} \log\left(\frac{\sqrt{(\alpha+1)(\beta+1)}}{2\pi}(\cos\theta)^{\alpha\cos^2\phi+\beta\sin^2\phi}\right) \\
&\quad= \frac{1}{2(\alpha+1)(\beta+1)}\sum_{s=1}^{M} f_s h_{si}((\beta+1) + 2(\alpha+1)(\beta+1)\cos^2\phi\log\cos\theta) \\
&\quad= \frac{\sum_{s=1}^{M} f_s h_{si}}{2(\alpha+1)} + \sum_{s=1}^{M} f_s h_{si}\cos^2\phi\log\cos\theta \\
&= 0 \Rightarrow \alpha = \frac{\sum_{s=1}^{M} f_s h_{si}}{-2\sum_{s=1}^{M} f_s h_{si}\cos^2\phi\log\cos\theta} - 1
\end{aligned}$$

The same goes for β. The final difference is a $\sin^2\phi$ in front of the logarithm.

1.3. Isotropic Beckmann Distribution

$$\begin{aligned}\frac{\partial}{\partial\lambda}\sum_{s=1}^{M} f_s h_{si}\log\left(\frac{e^{-(\tan^2\theta)/\lambda^2}}{\pi\lambda^2\cos^3\theta}\right) &= \frac{2}{\lambda^3}\sum_{s=1}^{M} f_s h_{si}(\tan^2\theta-\lambda^2)\\ &= \frac{2}{\lambda^3}\sum_{s=1}^{M} f_s h_{si}\tan^2\theta - \frac{2}{\lambda}\sum_{s=1}^{M} f_s h_{si}\\ = 0 \Rightarrow \lambda &= \sqrt{\frac{\sum_{s=1}^{M} f_s h_{si}\tan^2\theta}{\sum_{s=1}^{M} f_s h_{si}}}\end{aligned}$$

1.4. Anisotropic Beckmann Distribution

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\sum_{s=1}^{M} f_s h_{si}\log\left(\frac{\sqrt{\alpha\beta}e^{-(\tan^2\theta)(\alpha\cos^2\phi+\beta\sin^2\phi)}}{\pi\cos^3\theta}\right)\\ &= -\frac{1}{2\alpha\beta}\sum_{s=1}^{M} f_s h_{si}(2\alpha\beta\cos^2\phi\tan^2\theta-\beta)\\ &= -\sum_{s=1}^{M} f_s h_{si}\cos^2\phi\tan^2\theta + \frac{1}{2\alpha}\sum_{s=1}^{M} f_s h_{si}\\ = 0 \Rightarrow \alpha &= \frac{\sum_{s=1}^{M} f_s h_{si}}{2\sum_{s=1}^{M} f_s h_{si}\cos^2\phi\tan^2\theta}\end{aligned}$$

When comparing the isotropic and anisotropic Beckmann, the root is missing and the fraction inverted. This comes from the fact that we exchanged $1/\lambda^2$ with α and β, i.e. our parameters changed in the same way. So, the only real differences are the factors 2 and $\cos^2\phi$. The solution for β is the same except a $\sin^2\phi$ in front of the tangents.

1.5. Isotropic Gaussian A

The usual Gaussian on the sphere is the von-Mises-Fisher distribution. In the paper we introduced a variant of a Gaussian inspired by the Anisotropic-Spherical-Gaussian form [Xu et al. 2013]. For this model the parameter estimation looks as follows:

$$\begin{aligned}\frac{\partial}{\partial\lambda}\sum_{s=1}^{M} f_s h_{si}\log\left(\frac{\lambda\cos\theta}{\pi(1-e^{-\lambda})}e^{-\lambda\sin^2\theta}\right) &= -\sum_{s=1}^{M} f_s h_{si}\left(\sin^2\theta - \frac{e^{\lambda}-\lambda-1}{\lambda e^{\lambda}-\lambda}\right)\\ = 0 \Rightarrow \frac{e^{\lambda}-\lambda-1}{\lambda e^{\lambda}-\lambda} &= \frac{\sum_{s=1}^{M} f_s h_{si}\sin^2\theta}{\sum_{s=1}^{M} f_s h_{si}} \qquad (1)\end{aligned}$$

Similar to the quotient of Bessel functions in the estimation of vMF parameters (see [Banerjee et al. 2005]) we need an approximation to estimate λ. Let r be the number

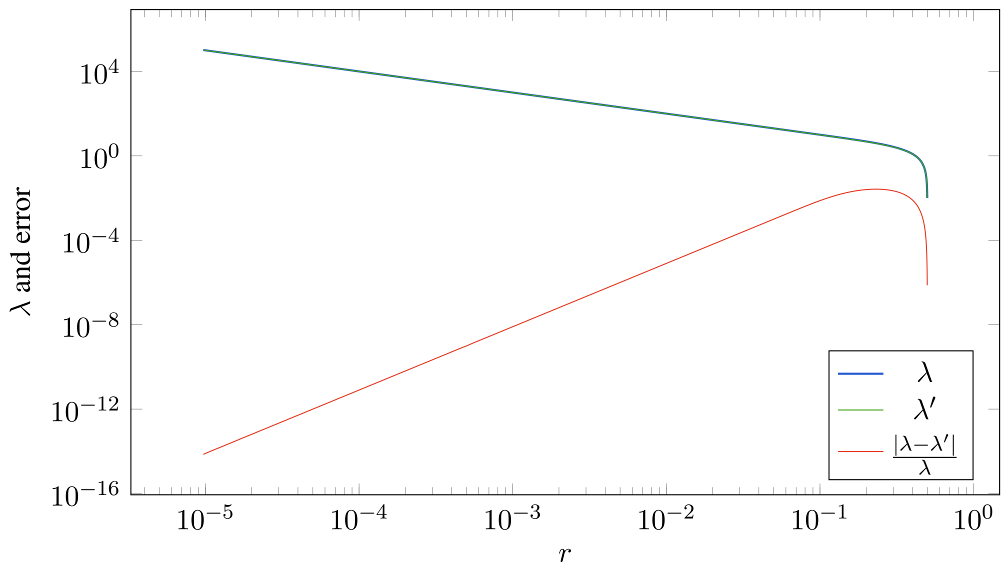

Figure 1. Comparison of real and estimated λ parameter for the Isotropic Gaussian A distribution. The bottom line shows the relative deviation $|\lambda - \lambda'|/\lambda$ where λ' is the approximated parameter.

obtained from the right sides quotient, then

$$\lambda' = \frac{1 - 8r^2}{r} \tag{2}$$

is a quite good approximation. We evaluated the approximation by computing the real r with the left side from Equation 1 and inserting that number into Equation 2. In the observed samples the maximum relative deviation was 2.64% at $\lambda = 4$. However, the relative errors are much small everywhere else (e.g. 0.16% at $\lambda = 0.5$ and $8 \cdot 10^{-9}$ at $\lambda = 1000$). See Figure 1 for details.

2. Polynomial Bases

This section provides the evaluated polynomials for Spherical Harmonics, Hemispherical Harmonics, Generalized $\mathcal{H}$ basis, Zernike's basis and Makhotkin's Basis for the first four bands. All polynomials are given in Cartesian and polar coordinates with $(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. A similar table for SH (5 bands) can be found in [Sloan 2008].

2.1. Spherical Harmonics (SH)

The basis functions

$$y_l^m(\theta,\varphi) = \begin{cases} \sqrt{2}K_l^{|m|} \cdot \cos|m|\varphi \cdot P_l^{|m|}(\cos\theta) & m > 0 \\ \sqrt{2}K_l^{|m|} \cdot \sin|m|\varphi \cdot P_l^{|m|}(\cos\theta) & m < 0 \\ \quad K_l^0 \ \cdot P_l^0(\cos\theta) & m = 0 \end{cases}$$

with $$K_l^m = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}$$

$$P_m^m(x) = (-1)^m(1-x^2)^{m/2}\prod_1^m(2m-1)$$

$$P_{m+1}^m(x) = x(2m+1)P_m^m$$

$$P_l^m(x) = x\left(\frac{2l-1}{l-m}\right)P_{l-1}^m - \left(\frac{l+m-1}{l-m}\right)P_{l-2}^m$$

yield the following polynomials:

l	m	Cartesian	Polar
0	0	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$
	-1	$-\frac{\sqrt{3}y}{2\sqrt{\pi}}$	$-\frac{\sqrt{3}\sin\varphi\sin\theta}{2\sqrt{\pi}}$
1	0	$\frac{\sqrt{3}z}{2\sqrt{\pi}}$	$\frac{\sqrt{3}\cos\theta}{2\sqrt{\pi}}$
	1	$-\frac{\sqrt{3}x}{2\sqrt{\pi}}$	$-\frac{\sqrt{3}\cos\varphi\sin\theta}{2\sqrt{\pi}}$
	-2	$\frac{\sqrt{15}yx}{2\sqrt{\pi}}$	$\frac{\sqrt{15}\sin 2\varphi\sin^2\theta}{4\sqrt{\pi}}$
	-1	$-\frac{\sqrt{15}yz}{2\sqrt{\pi}}$	$-\frac{\sqrt{15}\sin\varphi\cos\theta\sin^2\theta}{2\sqrt{\pi}}$
2	0	$\frac{\sqrt{5}(3z^2-1)}{4\sqrt{\pi}}$	$\frac{\sqrt{5}(3\cos^2\theta-1)}{4\sqrt{\pi}}$
	1	$-\frac{\sqrt{15}xz}{2\sqrt{\pi}}$	$-\frac{\sqrt{15}\cos\varphi\cos\theta\sin^2\theta}{2\sqrt{\pi}}$
	2	$\frac{\sqrt{15}(x^2-y^2)}{4\sqrt{\pi}}$	$\frac{\sqrt{15}\cos 2\varphi\sin^2\theta}{4\sqrt{\pi}}$
	-3	$-\frac{\sqrt{35}y(y^2-3x^2)}{4\sqrt{2\pi}}$	$-\frac{\sqrt{35}\sin 3\varphi\sin^3\theta}{4\sqrt{2\pi}}$
	-2	$\frac{2\sqrt{210}xyz}{4\sqrt{2\pi}}$	$\frac{\sqrt{210}\sin 2\varphi\cos\theta\sin^3\theta}{4\sqrt{2\pi}}$
	-1	$-\frac{\sqrt{21}y(5z^2-1)}{4\sqrt{2\pi}}$	$-\frac{\sqrt{21}\sin\varphi(5\cos^2\theta-1)\sin\theta}{4\sqrt{2\pi}}$
3	0	$\frac{\sqrt{14}z(5z^2-3)}{4\sqrt{2\pi}}$	$\frac{\sqrt{14}\cos\theta(5\cos^2\theta-3)}{4\sqrt{2\pi}}$
	1	$-\frac{\sqrt{21}x(5z^2-1)}{4\sqrt{2\pi}}$	$-\frac{\sqrt{21}\cos\varphi(5\cos^2\theta-1)\sin\theta}{4\sqrt{2\pi}}$
	2	$\frac{\sqrt{210}(x^2-y^2)z}{4\sqrt{2\pi}}$	$\frac{\sqrt{210}\cos 2\varphi\cos\theta\sin^3\theta}{4\sqrt{2\pi}}$
	3	$-\frac{\sqrt{35}x(x^2-3y^2)}{4\sqrt{2\pi}}$	$-\frac{\sqrt{35}\cos 3\varphi\sin^3\theta}{4\sqrt{2\pi}}$

2.2. Hemispherical Harmonics (HSH)

The HSH basis [Gautron et al. 2004] is similar to the SH basis above. The difference are a shift in the polynomials and a different normalization factor:

$$\tilde{P}_l^m(x) = P_l^m(2x-1)$$
$$\tilde{K}_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}.$$

Hence the evaluated polynomials are:

l	m	Cartesian	Polar
0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2\pi}}$
	-1	$-\frac{\sqrt{12}y\sqrt{z}}{\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{\sqrt{3}\sin\varphi\sqrt{\cos\theta-\cos^2\theta}}{\sqrt{2\pi}}$
1	0	$\frac{\sqrt{3}(2z-1)}{\sqrt{2\pi}}$	$\frac{\sqrt{3}(2\cos\theta-1)}{\sqrt{2\pi}}$
	1	$-\frac{\sqrt{12}x\sqrt{z}}{\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{\sqrt{3}\cos\varphi\sqrt{\cos\theta-\cos^2\theta}}{\sqrt{2\pi}}$
	-2	$\frac{4\sqrt{15}xyz}{\sqrt{2\pi}(z+1)}$	$\frac{2\sqrt{15}\sin 2\varphi(\cos\theta-\cos^2\theta)}{\sqrt{2\pi}}$
	-1	$-\frac{2\sqrt{15}y(2z-1)\sqrt{z}}{\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{2\sqrt{15}\sin\varphi(2\cos\theta-1)\sqrt{\cos\theta-\cos^2\theta}}{\sqrt{2\pi}}$
2	0	$\frac{\sqrt{5}(3(2z-1)^2-1)}{2\sqrt{2\pi}}$	$\frac{\sqrt{5}(3(2\cos\theta-1)^2-1)}{2\sqrt{2\pi}}$
	1	$-\frac{2\sqrt{15}x(2z-1)\sqrt{z}}{\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{2\sqrt{15}\cos\varphi(2\cos\theta-1)\sqrt{\cos\theta-\cos^2\theta}}{\sqrt{2\pi}}$
	2	$\frac{2\sqrt{15}(x^2-y^2)z}{\sqrt{2\pi}(z+1)}$	$\frac{2\sqrt{15}\cos 2\varphi(\cos\theta-\cos^2\theta)}{\sqrt{2\pi}}$
	-3	$-\frac{2\sqrt{35}y(3x^2-y^2)\sqrt{z^3}}{\sqrt{\pi}\sqrt{(z+1)^3}}$	$-\frac{2\sqrt{35}\sin 3\varphi\sqrt{(\cos\theta-\cos^2\theta)^3}}{\sqrt{\pi}}$
	-2	$\frac{4\sqrt{105}xy(2z-1)z}{\sqrt{2\pi}(z+1)}$	$\frac{4\sqrt{105}\sin 2\varphi(2\cos\theta-1)(\cos\theta-\cos^2\theta)}{\sqrt{2\pi}}$
	-1	$-\frac{\sqrt{42}y(5(2z-1)^2-1)\sqrt{z}}{2\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{\sqrt{42}\sin\varphi(5(2\cos\theta-1)^2-1)\sqrt{\cos\theta-\cos^2\theta}}{2\sqrt{2\pi}}$
3	0	$\frac{\sqrt{7}(20z^3-30z^2+12z-1)}{\sqrt{2\pi}}$	$\frac{\sqrt{7}(20\cos^3\theta-30\cos^2\theta+12\cos\theta-1)}{\sqrt{2\pi}}$
	1	$-\frac{\sqrt{42}x(5(2z-1)^2-1)\sqrt{z}}{2\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{\sqrt{42}\cos\varphi(5(2\cos\theta-1)^2-1)\sqrt{\cos\theta-\cos^2\theta}}{2\sqrt{2\pi}}$
	2	$\frac{2\sqrt{105}(x^2-y^2)(2z-1)z}{\sqrt{2\pi}(z+1)}$	$\frac{2\sqrt{105}\cos 2\varphi(2\cos\theta-1)(\cos\theta-\cos^2\theta)}{\sqrt{2\pi}}$
	3	$-\frac{2\sqrt{35}x(x^2-3y^2)\sqrt{z^3}}{\sqrt{\pi}\sqrt{(z+1)^3}}$	$-\frac{2\sqrt{35}\cos 3\varphi\sqrt{(\cos\theta-\cos^2\theta)^3}}{\sqrt{\pi}}$

2.3. Generalized $\mathcal{H}$ Basis

As stated in the paper the $\mathcal{H}$ Basis originally consists of only 6 functions [Habel and Wimmer 2010] which are marked in the upcoming table. The difference to the HSH basis is again another mapping of the argument passed to the Legendre polynomial.

$$\hat{P}_l^m(x) = P_l^m((1+s_l^m)\cdot x - s_l^m) \qquad \text{with} \qquad s_l^m = \sqrt{\frac{l-|m|}{l}}$$

The resulting polynomials are:

	l	m	Cartesian	Polar
$\mathcal{H}_4$	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2\pi}}$
$\mathcal{H}_4$		-1	$-\frac{\sqrt{3}y}{\sqrt{2\pi}}$	$-\frac{\sqrt{3}\sin\varphi\sin\theta}{\sqrt{2\pi}}$
$\mathcal{H}_4$	1	0	$\frac{\sqrt{3}(2z-1)}{\sqrt{2\pi}}$	$\frac{\sqrt{3}(2\cos\theta-1)}{\sqrt{2\pi}}$
$\mathcal{H}_4$		1	$-\frac{\sqrt{3}x}{\sqrt{2\pi}}$	$-\frac{\sqrt{3}\cos\varphi\sin\theta}{\sqrt{2\pi}}$
$\mathcal{H}_6$		-2	$\frac{\sqrt{15}xy}{\sqrt{2\pi}}$	$\frac{\sqrt{15}\sin 2\varphi\sin^2\theta}{2\sqrt{2\pi}}$
		-1	$-\frac{\sqrt{15}y(z+\sqrt{2}z-1)\sqrt{2\sqrt{2}z+3z+1}}{2\sqrt{2\pi}\sqrt{1+z}}$	$-\frac{\sqrt{15}\sin\varphi((1+\sqrt{2})\cos\theta-1)\sqrt{2-((1+\sqrt{2})\cos\theta-1)^2}}{2\sqrt{\pi}}$
	2	0	$\frac{\sqrt{5}(3(2z-1)^2-1)}{2\sqrt{2\pi}}$	$\frac{\sqrt{5}(3(2\cos\theta-1)^2-1)}{2\sqrt{2\pi}}$
		1	$-\frac{\sqrt{15}x(z+\sqrt{2}z-1)\sqrt{2\sqrt{2}z+3z+1}}{2\sqrt{2\pi}\sqrt{1+z}}$	$-\frac{\sqrt{15}\cos\varphi((1+\sqrt{2})\cos\theta-1)\sqrt{2-((1+\sqrt{2})\cos\theta-1)^2}}{2\sqrt{\pi}}$
$\mathcal{H}_6$		2	$\frac{\sqrt{15}(x^2-y^2)}{2\sqrt{2\pi}}$	$\frac{\sqrt{15}\cos 2\varphi\sin^2\theta}{2\sqrt{2\pi}}$
		-3	$-\frac{\sqrt{70}y(3x^2-y^2)}{4\sqrt{2\pi}}$	$-\frac{\sqrt{70}\sin 3\varphi\sqrt{\sin^3\theta}}{4\sqrt{2\pi}}$
		-2	$\frac{\sqrt{35}xy((6\sqrt{3}+10)z^2-2z-2)}{2\sqrt{2\pi}(z+1)}$	$\frac{\sqrt{35}\sin 2\varphi((1+\sqrt{3})\cos\theta-1)(3-((1+\sqrt{3})\cos\theta-1)^2)}{2\sqrt{2\pi}}$
		-1	$-\frac{\sqrt{21}y(5c^2-10c+4)\sqrt{1-(1-c)^2}}{4\sqrt{\pi}\sqrt{x^2+y^2}}$ *	$-\frac{\sqrt{21}\sin\varphi(5c^2-10c+4)\sqrt{1-(1-c)^2}}{4\sqrt{\pi}}$ *
	3	0	$\frac{\sqrt{7}(20z^3-30z^2+12z-1)}{\sqrt{2\pi}}$	$\frac{\sqrt{7}(20\cos^3\theta-30\cos^2\theta+12\cos\theta-1)}{\sqrt{2\pi}}$
		1	$-\frac{\sqrt{21}x(5c^2-10c+4)\sqrt{1-(1-c)^2}}{4\sqrt{\pi}\sqrt{x^2+y^2}}$ *	$-\frac{\sqrt{21}\cos\varphi(5c^2-10c+4)\sqrt{1-(1-c)^2}}{4\sqrt{\pi}}$ *
		2	$\frac{\sqrt{35}(x^2-y^2)((6\sqrt{3}+10)z^2-2z-2)}{2\sqrt{2\pi}(z+1)}$	$\frac{\sqrt{35}\cos 2\varphi((1+\sqrt{3})\cos\theta-1)(3-((1+\sqrt{3})\cos\theta-1)^2)}{2\sqrt{2\pi}}$
		3	$-\frac{\sqrt{70}x(x^2-3y^2)}{4\sqrt{2\pi}}$	$-\frac{\sqrt{70}\cos 3\varphi\sqrt{\sin^3\theta}}{4\sqrt{2\pi}}$

* $c = \left(\sqrt{\frac{2}{3}}+1\right)(1-z)$ * $c = \left(\sqrt{\frac{2}{3}}+1\right)(1-\cos\theta)$

2.4. Zernike's Basis

The Zernike's Basis [Zernike 1934; Koenderink et al. 1996] is defined on a disc and can be lifted to the sphere. The polynomials with our normalization are then defined as

$$Z_l^m(\theta,\varphi) = \begin{cases} \sqrt{\frac{l+1}{\pi}}\cos(|m|\varphi)\cdot R_l^{|m|}(\sqrt{2}\sin\frac{\theta}{2}) & m>0 \\ \sqrt{\frac{l+1}{\pi}}\sin(|m|\varphi)\cdot R_l^{|m|}(\sqrt{2}\sin\frac{\theta}{2}) & m<0 \\ \sqrt{\frac{l+1}{2\pi}}R_l^0(\sqrt{2}\sin\frac{\theta}{2}) & m=0 \end{cases}$$

$$R_l^m(x) = \sum_{k=0}^{(l-m)/2} \frac{(-1)^k (l-k)!}{k!\left(\frac{l+m}{2}-k\right)!\left(\frac{l-m}{2}-k\right)!}\cdot x^{l-2k}$$

yielding the evaluated polynomials

l	m	Cartesian	Polar
0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2\pi}}$
1	-1	$-\frac{2y}{\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{2\sqrt{2}\sin\varphi\sin\frac{\theta}{2}}{\sqrt{2\pi}}$
	1	$\frac{2x}{\sqrt{2\pi}\sqrt{1+z}}$	$\frac{2\sqrt{2}\cos\varphi\sin\frac{\theta}{2}}{\sqrt{2\pi}}$
	-2	$-\frac{2\sqrt{6}xy}{\sqrt{2\pi}(z+1)}$	$-\frac{2\sqrt{6}\sin 2\varphi\sin^2\frac{\theta}{2}}{\sqrt{2\pi}}$
2	0	$\frac{\sqrt{3}(1-2z)}{\sqrt{2\pi}}$	$\frac{\sqrt{3}(4\sin^2\frac{\theta}{2}-1)}{\sqrt{2\pi}}$
	2	$\frac{\sqrt{6}(x^2-y^2)}{\sqrt{2\pi}(z+1)}$	$\frac{2\sqrt{6}\cos 2\varphi\sin^2\frac{\theta}{2}}{\sqrt{2\pi}}$
3	-3	$-\frac{\sqrt{8}y(3x^2-y^2)}{\sqrt{2\pi}\sqrt{(z+1)^3}}$	$-\frac{8\sin 3\varphi\sin^3\frac{\theta}{2}}{\sqrt{2\pi}}$
	-1	$-\frac{\sqrt{8}y(1-3z)}{\sqrt{2\pi}\sqrt{z+1}}$	$-\frac{8\sin\varphi(3\sin^3\frac{\theta}{2}-\sin\frac{\theta}{2})}{\sqrt{2\pi}}$
	1	$\frac{\sqrt{8}x(1-3z)}{\sqrt{2\pi}\sqrt{z+1}}$	$\frac{8\cos\varphi(3\sin^3\frac{\theta}{2}-\sin\frac{\theta}{2})}{\sqrt{2\pi}}$
	3	$\frac{\sqrt{8}x(x^2-3y^2)}{\sqrt{2\pi}\sqrt{(z+1)^3}}$	$\frac{8\cos 3\varphi\sin^3\frac{\theta}{2}}{\sqrt{2\pi}}$
	-4	$-\frac{4\sqrt{10}xy(x^2-y^2)}{\sqrt{2\pi}(z+1)^2}$	$-\frac{4\sqrt{10}\sin 4\varphi\sin^4\frac{\theta}{2}}{\sqrt{2\pi}}$
	-2	$-\frac{\sqrt{10}xy(2-8z)}{\sqrt{2\pi}(z+1)}$	$-\frac{\sqrt{10}\sin 2\varphi(16\sin^4\frac{\theta}{2}-6\sin^2\frac{\theta}{2})}{\sqrt{2\pi}}$
4	0	$\frac{6\sqrt{5}(z^2-z+1)}{\sqrt{2\pi}}$	$\frac{\sqrt{5}(24\sin^4\frac{\theta}{2}-12\sin^2\frac{\theta}{2}+1)}{\sqrt{2\pi}}$
	2	$\frac{\sqrt{10}(x^2-y^2)(1-4z)}{\sqrt{2\pi}(z+1)}$	$\frac{\sqrt{10}\cos 2\varphi(16\sin^4\frac{\theta}{2}-6\sin^2\frac{\theta}{2})}{\sqrt{2\pi}}$
	4	$\frac{\sqrt{10}(x^4-6x^2y^2+y^4)}{\sqrt{2\pi}(z+1)^2}$	$\frac{4\sqrt{10}\cos 4\varphi\sin^4\frac{\theta}{2}}{\sqrt{2\pi}}$

2.5. Makhotkin's Basis

Last of all we have found the Makhotkin's basis [Makhotkin 1996] with following first few basis functions.

l	m	Cartesian	Polar
0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2\pi}}$
	-1	$\frac{\sqrt{6z}\,x}{\sqrt{\pi}\sqrt{z+1}}$	$\frac{\sqrt{6(\cos\theta-\cos^2\theta)}\cos\varphi}{\sqrt{\pi}}$
1	0	$\frac{3z-2}{\sqrt{2\pi}}$	$\frac{3\cos\theta-2}{\sqrt{2\pi}}$
	1	$\frac{\sqrt{6z}\,y}{\sqrt{\pi}\sqrt{z+1}}$	$\frac{\sqrt{6(\cos\theta-\cos^2\theta)}\sin\varphi}{\sqrt{\pi}}$
	-2	$\frac{\sqrt{30}z(x^2-y^2)}{\sqrt{\pi}(z+1)}$	$\frac{\sqrt{30}(\cos\theta-\cos^2\theta)\cos 2\varphi}{\sqrt{\pi}}$
	-1	$\frac{\sqrt{z}(10z-6)x}{\sqrt{\pi}\sqrt{z+1}}$	$\frac{\sqrt{\cos\theta-\cos^2\theta}(10\cos\theta-6)\cos\varphi}{\sqrt{\pi}}$
2	0	$\frac{10z^2-12z+3}{\sqrt{2\pi}}$	$\frac{10\cos^2\theta-12\cos\theta+3}{\sqrt{2\pi}}$
	1	$\frac{\sqrt{z}(10z-6)y}{\sqrt{\pi}\sqrt{z+1}}$	$\frac{\sqrt{\cos\theta-\cos^2\theta}(10\cos\theta-6)\sin\varphi}{\sqrt{\pi}}$
	2	$\frac{\sqrt{30}z2xy}{\sqrt{\pi}(z+1)}$	$\frac{\sqrt{30}(\cos\theta-\cos^2\theta)\sin 2\varphi}{\sqrt{\pi}}$
	-3	$\frac{\sqrt{140z^3}(x^3-3xy^2)}{\sqrt{\pi}\sqrt{(z+1)^3}}$	$\frac{\sqrt{140(\cos\theta-\cos^2\theta)^3}\cos 3\varphi}{\sqrt{\pi}}$
	-2	$\frac{\sqrt{15}(7z^2-4z)(x^2-y^2)}{\sqrt{\pi}(z+1)}$	$\frac{\sqrt{15}(\cos\theta-\cos^2\theta)(7\cos\theta-4)\cos 2\varphi}{\sqrt{\pi}}$
	-1	$\frac{\sqrt{30z}(7z^2-8z+30)x}{\sqrt{\pi}\sqrt{z+1}}$	$\frac{\sqrt{30(\cos\theta-\cos^2\theta)}(7\cos^2\theta-8\cos\theta+30)\cos\varphi}{\sqrt{\pi}}$
3	0	$\frac{35z^3-60z^2+30z-4}{\sqrt{2\pi}}$	$\frac{35\cos^3\theta-60\cos^2\theta+30\cos\theta-4}{\sqrt{2\pi}}$
	1	$\frac{\sqrt{30z}(7z^2-8z+30)y}{\sqrt{\pi}\sqrt{z+1}}$	$\frac{\sqrt{30(\cos\theta-\cos^2\theta)}(7\cos^2\theta-8\cos\theta+30)\sin\varphi}{\sqrt{\pi}}$
	2	$\frac{\sqrt{15}(7z^2-4z)2xy}{\sqrt{\pi}(z+1)}$	$\frac{\sqrt{15}(\cos\theta-\cos^2\theta)(7\cos\theta-4)\sin 2\varphi}{\sqrt{\pi}}$
	3	$\frac{\sqrt{140z^3}(3x^2y-y^3)}{\sqrt{\pi}\sqrt{(z+1)^3}}$	$\frac{\sqrt{140(\cos\theta-\cos^2\theta)^3}\sin 3\varphi}{\sqrt{\pi}}$

As stated in the main paper, the basis can be orthogonalized by multiplying $\sqrt{1+z}$ in Cartesian or $\sqrt{2\cos\theta}$ in Polar coordinates. Figure 2 shows the fitting performance of the two different bases. Additionally, we tried another modification of the basis which is still non-orthogonal, namely

$$\check{M}_l^m(\theta,\varphi) = \begin{cases} C_l^{|m|}\sin(|m|\varphi)\cdot J_l^{|m|}(\cos\theta) & m>0 \\ C_l^{|m|}\cos(|m|\varphi)\cdot J_l^{|m|}(\cos\theta) & m<=0 \end{cases},$$

for which the fitting performance is also given. From all three variants the orthogonalized one performs best. The SSIM index increases in all three cases, but only the two modified versions also decrease in RMSE. In general both modifications are very similar. Thus, the choice depends on the use case. For the orthogonalized variant the equator is always zero, whereas $\check{M}_l^m$ can have different values here.

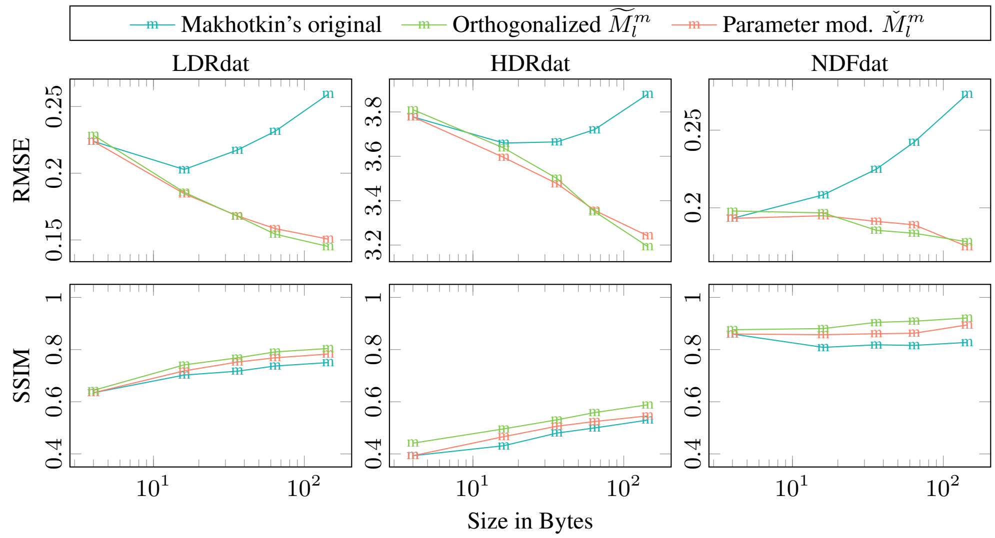

Figure 2. Error over size comparison for variants of Makhotkin's basis.

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